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Dynamic Asset Allocation with Commodities and Stochastic Interest Rates

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This research aims at finding an explicit investment policy with hedged variations of mixed bond-stock-commodity dynamic portfolio problems under a simple interest rate model and mean-reverting commodity prices. The findings suggest that the optimal allocation to a zero-coupon bond and a commodity is a combination of speculative terms and hedge terms as protection of change in interest rates and change in commodity market price of risk, respectively. The allocation to stocks, however, depends only on the speculative portfolio as there is no need for risk protection. The policy recommends a negative relationship between risk-aversion factor and riskier assets, stocks and commodities, while it proposes a positive relationship to that of zero-coupon bonds. This is consistent with the professional advice that investors who can tolerate more risk should invest more in riskier assets such as stocks and commodities. The paper also finds inverse relationships between commodity prices and positions in the stock and the commodity but no conclusions can be made regarding the direction of zero-coupon bond investment from a rise in commodity prices. The welfare loss due to neglect in commodity investment is also solved in explicit form. Nonetheless, the result should be verified with numerical examples so that one can determine how a bond-stock-commodity portfolio differs from a pure bond and stock portfolio.

Field of Research: Finance

1. Introduction

Commodities have played an important role as an alternative asset class for investors in recent years. Commodities have been emerging as an increasingly important class of assets and are claimed to have value-added effectiveness due to their diversification benefits. Ibbotson Associates (2006), exhibiting the correlation coefficients of annual total returns (1970-2004), provide intuitive evidence of the low correlation of commodities with traditional asset classes. Of the seven asset classes, treasury bills and commodities are the only two asset classes with negative average correlations to the other asset classes. In addition, commodities are positively correlated with inflation (see, for example, Erb and Harvey, 2006; Gorton and Rouwenhorst, 2006). There are some studies concerned with the diversification and inflation hedge effects of commodities especially research of short-term investment using the framework of Markowitz (1952).

Despite the rise of commodity investment, portfolios of most investors are still comprised mainly of traditional assets as stocks and bonds, so it is important to keep them in investment decisions. In fact, given the growing importance of commodities, it is interesting to set up a portfolio consisted of bonds, stocks, and commodities. To accomplish such an objective, this study combines characteristics from the models

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including term structure of interest rates as in Sørensen (1999) and Korn and Kraft (2001); the models of mean-reverting excess returns as in Kim and Omberg (1996), Wachter (2002), and Munk, Sørensen, and Vinther (2004); and the models including commodity asan alternative asset class as in Dai (2009). Two standard methods of investigation, dynamic programming and martingale method, can be used to determine optimal strategies. We will use the dynamic programming approach to solve the problem of allocating bonds, stocks, and commodities. In contrast to Dai (2009) who focuses only on stock and commodity selection, we present bonds into the portfolio selection which complicates the study. Yet, since bonds are one of the most important asset classes for both institutional and individual investors, introducing bonds increases practical applicability.

The paper aims at finding solutions of dynamic bond-stock-commodity allocation under a stochastic opportunity set of investors holding constant relative risk-aversion (CRRA) utility. It will also compare traditional advice from financial markets to results from the literature. For example, as debated in Campbell and Viceira (2002), two investment rules of thumb claim that (1) aggressive investors should hold stocks while conservative investors should hold only bonds; and (2) long-term investors invest more in stocks than short-term investors. This study will discuss whether those two beliefs hold up well when approached academically and how introduction of commodities affects these rules of thumb. Moreover, the paper considers relationships between movements in commodity prices and changes in investment strategies. In short, we discuss whether an increase in commodity prices leads to long or short positions in other assets. Finally, to understand the importance of having commodity assets in the portfolio, we examine wealth loss from excluding commodities from the portfolio.

The result is an explicit investment strategy with hedge variations in interest rates and the commodity market price of risk. In the optimal policy, the allocation to zero-coupon bonds and commodities is made for speculative or myopic purposes as well as for intertemporal hedging purposes. The optimal allocation to the stock depends on the spot commodity price and does not contain the hedge term compared to previous two assets. Positions in stocks and commodities, the riskier assets, have negative relationships with investor's risk tolerance, while positions in zero-coupon bonds, the less risky asset, have the opposite result. Despite finding the exact solution to the problem, there are at least three limitations in this study. The dynamic commodity price model is a one-factor model although recent studies have shown more options of two or multi-factor models that are more effective in illustrating commodity price movements. Moreover, the paper does not consider the correlation among commodity price returns, inflation, and the value of US dollars which are all of the debated topics in recent years. Finally, as we mainly discussed the result based on mathematical models, there are some inconclusive parts that require verification with numerically.

The rest of the paper is organized as follows. Section 2 discusses relevant literature. Section 3 sets up investment asset dynamics and examines an optimal asset allocation strategy. Section 4 analyzes such an optimal solution and investigates more in horizon effect. Welfare analysis will also be discussed in this section. Finally, section 5 concludes the paper.

2. Literature Review

Dynamic asset allocation within continuous-time economics environment has been studied intensively since Merton (1969, 1971). The study helps an investor find continuous-time portfolio strategies. The standard method of investigation is based on the stochastic control framework and the Hamilton-Jacobi-Bellman (HJB) equation, resulting in nonlinear equations which are typically hard to solve. Merton sets up the framework of dynamic portfolio problems for an investor who maximizes expected wealth utility at a given investment horizon. This groundbreaking work, however, has an impractical assumption of constant interest rates while it is well-known in financial markets that interest rates are not deterministic.

As to relax the above assumption and to introduce bonds as one of the investment choices, it is necessary to include the model of term structure of interest rate dynamics into portfolio problems. Sørensen (1999), Brennan and Xia (2000), and Korn and Kraft (2001) consider investment problems under stochastic interest rates of Vasicek (1977) type where an investor with CRRA utility can invest in a bank account, stocks, and bonds. They argue that the optimal investment strategy is a simple combination of a speculative term and a hedge term. While the former explains the need to optimize an immediate risk-return profile in a mean-variance framework, the latter describes how the investor protects stochastic behaviors of interest rates. Particularly, Brennan and Xia (2000) show that, consistent with popular recommendations, the bond-stock ratio has a positive relationship with the degree of risk-aversion. Using more advanced interest rate model assumptions, Munk and Sørensen (2004) investigate investment strategies similar to those assuming simple interest rate models. However, one of their key results is that the hedge portfolio is more sensitive to the current form of the term structure than to the specific dynamics of interest rates. Thus, not only is it easier to apply basic models such as the Vasicek model in our research, but it is also possibly sufficient in terms of correctness and practicality.

Another type of allocation problems related to this paper is the optimal portfolio with the stochastic market price of risk. As some empirical studies suggest evidence of mean reversion in stock returns, Kim and Omberg (1996) and Wachter (2002) achieve exact optimal investment strategies in a set-up with a constant risk-free interest rate r and stocks where the market price of risk is identical to the Sharpe ratio of the stock. Munk, Sørensen, and Vinther (2004) also find the exact optimal asset allocation strategy for a portfolio of bonds and stocks in a model featuring mean reversion in stock prices and inflation uncertainty. Their main result is the combination of speculative and hedge terms in the optimal investment policy as in aforementioned papers. Instead of assuming mean reversion in stock prices, we will apply the stochastic characteristics of market price of risk into commodity prices as in Schwartz (1997).

As mentioned earlier about the importance and popularity of commodities, in the literature, it is still debatable whether one should include commodities in the portfolio and, if included, how commodities affect the total wealth. Research in commodity investment has commonly been founded on the performance of commodity futures since most commodities traded in financial markets are in derivative forms. Most existing studies on commodity investment apply the one-period mean-variance optimization framework of Markowitz (1952). In that kind of myopic framework, the main hypotheses examined by these studies are whether commodity investment gives a positive risk premium, correlates with other assets, and is capable of hedging against inflation. For example, Erb and Harvey (2006) find that some security characteristics and portfolio strategies provides a positive risk premiums. Conover, Jensen, Johnson, and Mercer (2010) also show that the total portfolio can be benefited by inclusion of commodities. Specifically, an equity portfolio with commodity exposures performs better during periods of high inflation.

However, much less research efforts have been devoted to long-term allocation strategy. The closest literature to the present paper is the work by Dai (2009), studying dynamic asset allocation using the Martingale approach with a focus on introducing commodities into portfolio management. As the same objectives of those who study allocation with stochastic interest rates, he solves for the exact solution of optimal portfolio and consumption strategies with the advent of commodities as a new asset class. The conclusion is that such policies are a combination of a speculative term and a hedge term. He also finds the wealth loss due to disregard in commodity investment. The main difference between Dai (2009) and our work is that while Dai uses the Martingale method as a means of finding the result, the present research paper applies the traditional dynamic programming approach to investigate the strategy. Moreover, Dai's portfolio includes stocks and commodities while this study includes bonds, stocks, and commodities; thus adding more complication in a search for an optimal investment strategy. To provide thorough details of the present article, the next section will set up related mathematical models and solve the problem of dynamic asset allocation.

3. Investment Asset Dynamics and Optimal Asset Allocation

In this section we introduce the investment asset dynamics and follow up by solving for the optimal asset allocation.

3.1 Investment Asset Dynamics

The interest rate dynamics are explained by Ornstein-Uhlenbeck process as in Vasicek (1977).

$$\mathrm{d}r_t = \kappa(\bar{r} - r_t)\mathrm{d}t - \sigma_r \mathrm{d}Z_{rt} \tag{1}$$

where $\bar{r}$ indicates the long-run mean of the interest rate, κ denotes the degree of mean reversion, σ_r is the volatility of the interest rate, and Z_{rt} is a standard Brownian motion. Such a process leads to a zero-coupon bond price with maturity $\bar{T}$ given by

$$B_t = \exp\left\{-a\left(\bar{T} - t\right) - b\left(\bar{T} - t\right)r_t\right\} \tag{2}$$

where

$$a(\tau) = \left[\bar{r} + \frac{\sigma_r \lambda_1}{\kappa} - \frac{1}{2}\left(\frac{\sigma_r}{\kappa}\right)^2\right]\left(\tau - b(\tau)\right) + \frac{\sigma_r^2}{4\kappa}\left(b(\tau)\right)^2,\ b(\tau) = \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa\tau}\right)$$

with the constant parameter, λ_1, is the premium on interest rate risk.

Using Ito's lemma, the dynamics of the bond price, $B_t^{\bar{T}}$, can be described by a stochastic differential equation in the form

$$\mathrm{d}B_t^{\bar{T}} = B_t^{\bar{T}} \left[\left(r_t + \sigma_B(r_t,t)\lambda_1 \right)\mathrm{d}t + \sigma_B(r_t,t)\mathrm{d}Z_{rt} \right] \quad (3)$$

where $\sigma_B(r_t,t) = \sigma_r b\left(\bar{T} - t\right)$ is the sensitivity term of the zero-coupon bond price. The dynamics of a stock price, S_t, are assumed as the following stochastic differential equation

$$\mathrm{d}S_t = S_t \left[\left(r_t + \sigma_s \psi \right) dt + \rho_{rs}\sigma_s \mathrm{d}Z_{rt} + \sqrt{1-\rho_{rs}^2}\,\sigma_s \mathrm{d}Z_{st} \right] \quad (4)$$

where σ_s describes the stock volatility, ψ denotes the stock market price of risk, and the product of these two parameters expresses expected excess return from equity investment. The correlation between returns in stocks and interest rates is denoted by ρ_{rs}. Z_{st} is another standard Brownian motion and independent of Z_{rt}. The spot commodity price, C_t, is assumed to follow the one-factor model in Schwartz (1997) given by

$$\mathrm{d}C_t = \theta\left(\mu_c - \ln C_t \right) C_t \mathrm{d}t + \sigma_c C_t \mathrm{d}\hat{Z}_{ct} \quad (5)$$

where μ_c is the long–run mean of the spot price, θ denotes the degree of mean reversion, σ_c explains the commodity volatility, and $\hat{Z}_{ct}$ is a standard Brownian motion. The model is popular for modeling energy and agricultural commodities and aims at introducing mean reversion to the long-run mean, μ_c.

Under risk-neutral measure, it is possible to find the futures price of such a commodity with the following market price of risk

$$\lambda_{ct} = \lambda_{c1} + \lambda_{c2} X_t,\ X_t = \ln C_t \quad (6)$$

As discussed in Dai (2009), the above equation is inspired by empirical evidence that expected returns are time-varying and can be predicted by some instrumental variables such as spot commodity prices themselves. Note that this model is reduced to the one-factor model of Schwartz (1997) if $\lambda_{c2} = 0$.

Also, following Dai (2009), the commodity price process may be adapted to the following process

$$\mathrm{d}V_t = V_t \left[\left(r_t + \sigma_c \lambda_{ct} \right)\mathrm{d}t + \sigma_c \mathrm{d}\hat{Z}_{ct} \right] \quad (7)$$

where V_t is the self-financing portfolio characterized as a new asset class of commodities.

With this process, it is possible to set up the dynamics of three assets, bonds, stocks, and commodities consistent with asset allocation problems of Merton (1969, 1971). The dynamics of three assets can be expressed in terms of the following matrix

$$\mathrm{d}\underline{P}_t = \begin{bmatrix} \mathrm{d}B_t^{\bar{T}} \\ \mathrm{d}S_t \\ \mathrm{d}V_t \end{bmatrix} = \mathrm{diag}(P_t)\left[\left(r\underline{1} + \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\underline{\lambda}(\lambda_{3t})\right)\mathrm{d}t + \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\mathrm{d}\underline{Z}_t\right] \tag{8}$$

where

$$\underline{\lambda}(\lambda_{3t}) = \left[\lambda_1 \quad \lambda_2 \quad \lambda_{3t}\right]^{\mathrm{T}}$$

$$\underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right) = \begin{bmatrix} \sigma_B(r_t, t) & 0 & 0 \\ \rho_{rs} & \bar{\rho}\sigma_s & 0 \\ \rho_{cr} & \rho_{sc} & \tilde{\rho}\sigma_s \end{bmatrix}, \ \bar{\rho} = \sqrt{1-\rho_{rs}^2}, \ \tilde{\rho} = \sqrt{1-\rho_{cr}^2-\rho_{sc}^2}$$

\and where parameters with one underline are vectors and parameters with two underlines denotes matrices. The vector of market price of risk is composed of two constants, λ_1 and λ_2, with respect to Z_{rt} and Z_{st} respectively; and another state variable, λ_{3t}, with respect to Z_{ct}. Note that the Brownian motion Z_{ct} is independent of Z_{rt} and Z_{st}. The parameter λ_{3t} is stochastic and dependent on the commodity price. λ_2 and λ_{3t} can be inferred from the above price dynamics as

$$\lambda_2 = \left(\psi - \rho_{\mathrm{rs}}\lambda_1\right)/\bar{\rho} \tag{9}$$

and

$$\lambda_{3t} = \left(\lambda_{ct} - \rho_{cr}\lambda_1 - \rho_{sc}\lambda_2\right)/\tilde{\rho} \tag{10}$$

As λ_{ct} depends on the commodity price, The market price of risk, λ_{3t}, can be expressed in the following dynamics

$$\mathrm{d}\lambda_{3t} = \theta\left(\bar{\lambda}_{3t} - \lambda_{3t}\right)\mathrm{d}t + \left(\lambda_2\sigma_c/\tilde{\rho}\right)\left[\rho_{cr} \quad \rho_{sc} \quad \tilde{\rho}\right]\mathrm{d}\underline{Z}_t \tag{11}$$

where

$$\bar{\lambda}_{3t} = \left(1/\tilde{\rho}\right)\left(\lambda_{c2}\alpha + \lambda_{c1} - \rho_{cr}\lambda_1 - \rho_{sc}\lambda_2\right)$$

The stochastic characteristics of the commodity market price of risk and the interest rate as mentioned earlier generates a stochastic investment opportunity set effect in hedging terms. The optimal investment strategy will be different from the case with static mean-variance framework. Two state variables, r_t and λ_{3t}, can be written in the following matrix

$$\begin{bmatrix} \mathrm{d}r_t \\ \mathrm{d}\lambda_{3t} \end{bmatrix} = \begin{bmatrix} \kappa\left(\bar{r} - r_t\right) \\ \theta\left(\bar{\lambda}_{3t} - \lambda_{3t}\right) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} -\sigma_r & 0 & 0 \\ \frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{cr} & \frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{sc} & \lambda_{c2}\sigma_c \end{bmatrix} \mathrm{d}\underline{Z}_t \qquad (12)$$

or

$$\mathrm{d}\underline{x}_t = \underline{m}\left(r, \lambda_{3t}\right)\mathrm{d}t + \underline{\underline{v}}\left(r, \lambda_{3t}\right)^{\mathrm{T}} \mathrm{d}\underline{Z}_t$$

3.2 Optimal Asset Allocation

Following Munk (2010), the wealth process, W_t, can be expressed as

$$\mathrm{d}W_t = W_t\left[r_t + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right) \underline{\lambda}(\lambda_{3t})\right]\mathrm{d}t + W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\mathrm{d}\underline{Z}_t \qquad (13)$$

where $\underline{\pi}_t$ is the three-dimensional vector of investment portion at time t in the portfolio consisted of bonds, stocks, and commodities. The remains of this wealth, $1 - \pi_B - \pi_S - \pi_V$, is invested in the risk-free asset.

An investor is assumed to maximize utility from the terminal wealth, W_T, with respect to a power utility function. The indirect utility function is given by

$$J\left(W, r, \lambda_{3t}, t\right) = \sup E_{W,r,\lambda_{3t},t}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \qquad (14)$$

where the parameter $\gamma > 0$ describes the risk tolerance level of an investor.

The Hamilton-Jacobi-Bellman equation associated with the above dynamic optimization problem has the form

$$\begin{aligned} 0 = {} & W J_W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right) \underline{\lambda}(\lambda_{3t}) + \frac{1}{2} J_{WW} W^2 \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right) \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)^{\mathrm{T}} \underline{\pi}_t + J_t \\ & + W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right) \underline{\underline{v}}\left(r, \lambda_{3t}\right) J_{W\underline{x}} + rWJ_W + J_{\underline{x}}^{\mathrm{T}} \underline{m}\left(r, \lambda_{3t}\right) + \frac{1}{2}\mathrm{tr}\left(J_{\underline{xx}} \underline{\underline{\Sigma}}\left(r, \lambda_{3t}\right)\right) \end{aligned} \qquad (15)$$

Where

$$\underline{\underline{\Sigma}}\left(r, \lambda_{3t}\right) = \underline{\underline{v}}\left(r, \lambda_{3t}\right)^{\mathrm{T}} \underline{\underline{v}}\left(r, \lambda_{3t}\right)$$

The first order condition with respect to the investment strategy provides the following form

$$\underline{\Pi} = -\frac{J_W}{WJ_{WW}}\left(\underline{\underline{\sigma}}\left(r_t,\lambda_{3t},t\right)^{\mathrm{T}}\right)^{-1}\underline{\lambda}(\lambda_{3t}) - \frac{J_{W\underline{x}}}{WJ_{WW}}\left(\underline{\underline{\sigma}}\left(r_t,\lambda_{3t},t\right)^{\mathrm{T}}\right)^{-1}\underline{\underline{v}}\left(r,\lambda_{3t}\right) \qquad (16)$$

The optimal asset allocation of a power utility investor is stated in the following theorem.

Theorem 1: *The indirect utility of wealth function of a CRRA investor is given by*

$$J\left(W,r,\lambda_{3t},t\right) = \frac{1}{1-\gamma}\left(W\mathrm{e}^{A_0(T-t)+A_1(T-t)r+A_2(T-t)\lambda_{3t}+A_3(T-t)\lambda_{3t}^2}\right)^{1-\gamma} \qquad (17)$$

where

$$A_0(\tau) = \frac{1}{2\gamma}\left(\lambda_1^2+\lambda_2^2\right)\tau + \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_1\right)\int_0^\tau A_1(s)\mathrm{d}s$$
$$+\left(\theta\bar{\lambda}_{3t} - \frac{\gamma-1}{\gamma}g\right)\int_0^\tau A_2(s)\mathrm{d}s + \frac{1}{2}\sigma_*^2\int_0^\tau A_3(s)\mathrm{d}s - \frac{\gamma-1}{2\gamma}\sigma_r^2\int_0^\tau A_1^2(s)\mathrm{d}s$$
$$+\frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\sigma_r\rho_{cr}\int_0^\tau A_1(s)A_2(s)\mathrm{d}s + \sigma_*^2\int_0^\tau A_2^2(s)\mathrm{d}s$$

$$A_1(\tau) = \frac{1}{\kappa}\left(1-\mathrm{e}^{-\kappa\tau}\right),\ A_2(\tau) = \left[\frac{2af\left(\mathrm{e}^{d\tau/2}-1\right)^2}{d\left(\mathrm{e}^{d\tau}-1\right)}\right]A_3(\tau),\ A_3(\tau) = \frac{2a\left(e^{d\tau}-1\right)}{\left(b+d\right)\left(e^{d\tau}-1\right)+2d},$$

with

$$a = \frac{1}{\gamma},\ b = 2\left(\theta + \frac{\gamma-1}{\gamma}\lambda_{c2}\sigma_c\right),\ c = -\frac{\gamma-1}{\gamma}\sigma_*^2,\ d = \sqrt{b^2-4ac},$$

$$\sigma_*^2 = \left(\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{cr}\right)^2 + \left(\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{sc}\right)^2 + \left(\lambda_{c2}\sigma_c\right)^2,$$

$$f = \theta\bar{\lambda}_{3t} - \frac{\gamma-1}{\gamma}\left(g + \frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\sigma_r\rho_{cr}A_1(\tau)\right),\ g = \frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\left(\rho_{cr}\lambda_1 + \rho_{sc}\lambda_2\right)$$

The vector of optimal risk asset allocations at time t *is given by*

$$\underline{\Pi} = \begin{bmatrix}\pi_B\\ \pi_S\\ \pi_V\end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma\sigma_B(r,t)}\left[\lambda_1 - \frac{\lambda_2\rho_{rs}}{\bar{\rho}} - \frac{\lambda_{3t}}{\bar{\rho}\tilde{\rho}}\left(\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}\right)\right] + \left(\frac{\gamma-1}{\gamma}\right)\frac{\sigma_r A_1(\tau)}{\sigma_B(r,t)} \\ \frac{1}{\gamma\sigma_S}\left(\frac{\lambda_2}{\bar{\rho}} - \frac{\lambda_{3t}\rho_{sc}}{\bar{\rho}\tilde{\rho}}\right) \\ \frac{\lambda_{3t}}{\gamma\tilde{\rho}\sigma_C} - \left(\frac{\gamma-1}{\gamma}\right)\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2(\tau) + A_3(\tau)\lambda_{3t}\right] \end{bmatrix} \qquad (18)$$

From the optimal strategy, there exist speculative terms and hedge terms as in previous studies. When risk-aversion factor, $\gamma > 1$, increases, positions in the stock decreases while the position in a zero-coupon bond increases. This is the same as the professional advice that investors who can tolerate less risk should invest more in less riskier assets such as the zero-coupon bond. Positions in the commodity, however, do not necessarily have an exact movement related to the risk-aversion factor. Also, we notice that allocation to the stock is independent from the investment horizon. This contradicts with traditional advice that the stock weight should increase with investment horizon. Basically, hedge terms in the zero-coupon bond and the commodity explain the hedge against change in interest rates and change in commodity market price of risk, respectively. Next section will discuss the theorem in more details including how investment horizon affects the portion of commodities in the portfolio.

4. Findings and Discussions

This section analyzes the strategy in Theorem 1 and considers wealth loss due to excluding commodities from the portfolio.

4.1 Allocation Analysis and Investment Horizon Effect

In this subsection we investigate the result from the optimal investment strategy obtained from the previous section. For simplicity, each component of wealth fraction will be discussed following with an argument in horizon effect in commodities.

Zero-coupon bond allocation

$$\pi_B = \frac{1}{\gamma \sigma_B(r,t)}\left[\lambda_1 - \frac{\lambda_2 \rho_{rs}}{\bar{\rho}} - \frac{\lambda_{3t}}{\bar{\rho}\tilde{\rho}}\left(\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}\right)\right] + \left(\frac{\gamma - 1}{\gamma}\right)\frac{\sigma_r A_1(\tau)}{\sigma_B(r,t)} \quad (19)$$

With the advent of commodity as an asset choice, there exists the term $-\lambda_{3t}\left(\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}\right)/\bar{\rho}\tilde{\rho}$ which could be either positive or negative depending on correlation factors, ρ_{cr}, ρ_{rs}, and ρ_{sc}. Regularly, empirical studies as in Ibbotson Associates (2006) suggest positive ρ_{rs} and negative in ρ_{sc} and ρ_{cr}. Also, Dai's (2009) finding that λ_{c2} is significantly less than zero, along with the empirical test of Schwartz (1997) concluding that the spot commodity price has significantly negative effect on the risk premium, suggests an inverse relationship between the commodity price, C_t, and the market price of risk, λ_{3t}. However, we still cannot ascertain the direction of zero-coupon bond caused by arise in the commodity price since the sign of the term $\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}$ also depends on magnitudes of each constant.

The hedge term suggests that, in the long run, investment in the zero-coupon bond depends on the interest rate volatility and the term $A_1(\tau)$ but not on the commodity. If the allocation in bonds is the zero-coupon maturing at the end of the investment horizon, T, we will obtain the equality $A_1(\tau) = \sigma_B(r,t)$ and the hedge term will reduce to $(\gamma - 1)/\gamma$ which no longer depends on time. By this assumption, it is clear that the hedge position of a more risk-averse investor $(\gamma > 1)$ is positive whereas a less risk-averse investor $(\gamma < 1)$ is negative.

Stock Allocation

$$\pi_S = \frac{1}{\gamma\sigma_S}\left(\frac{\lambda_2}{\bar{\rho}} - \frac{\lambda_{3t}\rho_{sc}}{\bar{\rho}\tilde{\rho}}\right) \tag{20}$$

As shown in (16) that part of the hedge term is created by the derivative $J_{W\underline{x}}$ while there are no state variables related to the stock compared to those of other assets. Stock allocation, thus, restricts only in the speculative term as there are no risks to hedge as in bonds as explained earlier or commodities which will be discussed later. Note that Munk et al. (2004) assume the mean reversion in stocks and find the hedge term in stock allocation. By examining (20), it is clear from the equation that if ρ_{sc} is greater than zero, the stock investment increases when commodity price increases. However, if ρ_{sc} is less than zero, the investment in stock has negative relationship with the commodity price; this case is more suggested empirically. It may be interpreted that when the commodity price reduces, this suggests in the increase in the stock price; and investors, therefore, should invest in stock.

Commodity Allocation

$$\pi_V = \frac{\lambda_{3t}}{\gamma\tilde{\rho}\sigma_C} - \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2(\tau) + A_3(\tau)\lambda_{3t}\right] \tag{21}$$

The result is similar to Munk et al. (2004) and Dai (2009) that the hedge term includes both $A_2(\tau)$ and $A_3(\tau)$. Focusing only on the first term, the speculative portfolio, it can be implied that, with λ_{c2} less than zero, there is a negative relationship between the commodity price and the speculative term.

Next, we concentrate on the hedge term and recall the value of $A_3(\tau)$. The fact that d is greater than b in Theorem 1 indicates the positivity of this term. This, therefore, leads to an inverse relationship between the commodity price and the hedge portfolio. All in all, it is certain to settle that the investor should invest in commodities when the commodity price decreases and reduce the portion when the commodity price increases.

Horizon Effect in Commodities

As described above that allocation to the stock is independent from the investment horizon. However, it remains unclear how investment horizon affects positions in the commodity. By differentiating the commodity wealth fraction with respect to time horizon, we obtain the following derivative

$$\frac{\partial \pi_V}{\partial \tau} = -\frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2'(\tau) + A_3'(\tau)\lambda_{3t}\right] \tag{22}$$

It can be shown that $A_3'(\tau)$ is always positive while $A_2'(\tau)$ can be either positive or negative depending on parameters. Thus, it is inconclusive whether the investment horizon has positive or negative effect in commodity investment. However, if λ_{3t} is high enough, the derivative will be positive and lead to a higher commodity allocation for a longer term investor. Further empirical studies are suggested to examine the horizon effect in commodities and to investigate investment rules of thumb.

4.2 Welfare Analysis

This subsection studies the wealth loss due to disregard in commodity investment. The loss is assumed as the percentage L of extra initial wealth that is necessary to bring the investor to the same utility level as the investor considering investment in a commodity.

$$J^{NC}\left(W(1+L), r, \lambda_{3t}, t\right) = J\left(W, r, \lambda_{3t}, t\right) = \frac{1}{1-\gamma}\left(W e^{H(r,\lambda_{3t},\tau)}\right)^{1-\gamma} \tag{23}$$

Therefore, we obtain

$$L = \exp\left\{H(r,\lambda_{3t},\tau) - H^{NC}(r,\tau)\right\} - 1 \tag{24}$$

where $H(r,\lambda_{3t},\tau) = A_0(\tau) + A_1(\tau)r + A_2(\tau)\lambda_{3t} + A_3(\tau)\lambda_{3t}^2$
and $H^{NC}(r,\tau) = A_0^{NC}(\tau) + A_1^{NC}(\tau)r$ is the function with no commodity consideration.

From Munk (2010), it follows that $A_1^{NC}(\tau) = A_1(\tau)$ and

$$A_0^{NC}(\tau) = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right)\tau + \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_1\right)\int_0^\tau A_1(s)\mathrm{d}s - \frac{\gamma-1}{2\gamma}\sigma_r^2\int_0^\tau A_1^2(s)\mathrm{d}s \tag{25}$$

Inserting $H(r,\lambda_{3t},\tau)$ and $H^{NC}(r,\tau)$ into the wealth loss equation, we obtain

$$\begin{aligned} L = \exp\Bigg\{ & A_2(\tau)\lambda_{3t} + \frac{1}{2}A_3(\tau)\lambda_{3t}^2 + \left(\theta\bar{\lambda}_{3t} - \frac{\gamma-1}{\gamma}\right)\int_0^\tau A_1(s)\mathrm{d}s + \frac{1}{2}\sigma_*^2\int_0^\tau A_3(s)\mathrm{d}s \\ & + \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\sigma_r\rho_{cr}\int_0^\tau A_1(s)A_2(s)\mathrm{d}s + \sigma_*^2\int_0^\tau A_2^2(s)\mathrm{d}s \Bigg\} - 1 \end{aligned} \tag{26}$$

It can be seen from the equation that with very high or very low commodity prices, the square of market price of risk will be high, leading to an increase in the welfare loss. On the other hand, the standard level of commodity values can reduce wealth loss. This may be reckoned as the opportunity loss from extreme movement in commodity prices.

5. Conclusions

This study has aimed at examining the bond-stock-commodity portfolio in the framework of dynamic asset allocation as in Merton (1969). The commodity market price of risk is stochastic and dependent on a spot commodity price, while the spot price itself has the property of mean reversion. As it is well-documented that interest rates are stochastic, the paper assumes the simple Vasicek interest rate model along with the mean reversion in the spot commodity price. The study has aimed at finding an optimal investment strategy for a portfolio including zero-coupon bonds, stocks, and commodities using the dynamic programming approach.

The optimal investment policy was derived for an investor who is concerned with terminal wealth. Closed-form expressions were obtained for the optimal strategies and the utility losses of excluding the commodity from the financial decisions. In the optimal policy, the allocation to zero-coupon bonds and commodities were made for speculative or myopic purposes, as well as for intertemporal hedging purposes. The optimal allocation to the stock is solely speculative and dependent on the spot commodity price. Positions in stocks and commodities, the riskier assets, have negative relationships with investor's risk tolerance, while positions in zero-coupon bonds, the less risky assets, have the opposite result. This is consistent with the professional advice that investors who can accept more risk should invest more in riskier assets.

Assuming that the spot commodity price has a significant negative effect on the risk premium as in Schwartz (1997) and Dai (2009), no conclusions could be made regarding the direction of zero-coupon bond investment resulting from a rise in commodity prices. However, it may be concluded that there are inverse relationships between commodity prices and positions in stocks and commodities in the portfolio. In brief, one should invest in stocks and commodities when commodity prices decrease and divest such positions when commodity prices increase. The welfare loss due to neglect in commodity investment is also solved in explicit form. The result implies that extreme movement in commodity prices may lead to more wealth loss compared to stable commodity prices. In our further empirical studies will determine the relation between commodity prices and investment in zero-coupon bonds as well as examine whether the wealth loss creates a huge difference between bond-stock-commodity portfolio and pure bond and stock portfolio in our further studies.

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Appendix: Proof of Theorem 1

Substituting the candidate optimal value of investment strategy into (15) and gathering terms, we obtain the second order PDE

$$\begin{aligned} 0 = J_t + rWJ_W - \frac{1}{2}\frac{J_W^2}{J_{WW}}\left\|\underline{\lambda}\left(\lambda_{3t}\right)\right\|^2 + J_{\underline{x}}^{\mathrm{T}}\underline{m}\left(r,\lambda_{3t}\right) + \frac{1}{2}\mathrm{tr}\left(J_{\underline{xx}}\underline{\underline{\Sigma}}\left(r,\lambda_{3t}\right)\right) \\ -\underline{\lambda}(\lambda_{3t})^{\mathrm{T}}\underline{\underline{v}}\left(r,\lambda_{3t}\right)\frac{J_W J_{W\underline{x}}}{J_{WW}} - \frac{1}{2J_{WW}}J_{W\underline{x}}^{\mathrm{T}}\underline{\underline{v}}\left(r,\lambda_{3t}\right)^{\mathrm{T}}\underline{\underline{v}}\left(r,\lambda_{3t}\right)J_{W\underline{x}} \end{aligned} \tag{27}$$

If this PDE has a solution, $J\left(W,r,\lambda_{3t},t\right)$, such that the strategy $\underline{\pi}_t$ is feasible, then the strategy is optimal and that the function $J\left(W,r,\lambda_{3t},t\right)$ is equal to the indirect utility function. Interested readers may consult Øksendal (2003) and Björk (2009) for more details. A guess solution to the PDE is given by

$$J\left(W,r,\lambda_{3t},t\right) = \frac{1}{1-\gamma}\left(W\,\mathrm{e}^{A_0(T-t)+A_1(T-t)r+A_2(T-t)\lambda_{3t}+A_3(T-t)\lambda_{3t}^2}\right)^{1-\gamma} \tag{28}$$

Substituting the relevant derivatives of $J\left(W,r,\lambda_{3t},t\right)$ into (27) and simplifying, we obtain that J will be a solution if the function H solves the PDE

$$\begin{aligned} 0 = r + \frac{1}{2\gamma}\left\|\underline{\lambda}\left(\lambda_{3t}\right)\right\|^2 - H_\tau\left(r,\lambda_{3t},\tau\right) + \left[\underline{m}\left(r,\lambda_{3t}\right) - \frac{\gamma-1}{\gamma}\underline{\underline{v}}\left(r,\lambda_{3t}\right)^{\mathrm{T}}\underline{\lambda}\left(\lambda_{3t}\right)\right]^{\mathrm{T}} H_{\underline{x}} \\ + \frac{1}{2}\mathrm{tr}\left(\underline{\underline{\Sigma}}\left(r,\lambda_{3t}\right)H_{\underline{xx}}\right) - \frac{\gamma-1}{2\gamma}H_{\underline{x}}^{\mathrm{T}}\underline{\underline{\Sigma}}\left(r,\lambda_{3t}\right)H_{\underline{x}} \end{aligned} \tag{29}$$

Guess that equation is in the form

$$H\left(r,\lambda_{3t},\tau\right) = A_0(\tau) + A_1(\tau)r + A_2(\tau)\lambda_{3t} + A_3(\tau)\lambda_{3t}^2 \tag{30}$$

Finding the relevant derivatives and inserting them into (29), we obtain an equation linear in the interest rate and quadratic in the commodity market price of risk. Its four coefficients must be zero, resulting in the following system of ODEs.

$$A_1'(\tau) = 1 - \kappa A_1(\tau),\ A_1(0) = 0 \tag{31}$$

$$A_3'(\tau) = a - bA_3(\tau) + cA_3(\tau)^2,\ A_3(0) = 0 \tag{32}$$

$$A_2'(\tau) = fA_3(\tau) - \left[\frac{1}{2}b - cA_3(\tau)\right]A_2(\tau),\ A_2(0) = 0 \tag{33}$$

where $a, b, c, d,$ and f are presented in Theorem 1. Solving the above PDEs, we obtain $A_1(\tau)$, $A_2(\tau)$, and $A_3(\tau)$ as also shown in the theorem. Plugging them into the optimal strategy, we achieve the optimal investment strategy.